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# Additivity of Jordan maps on standard operator algebras

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**Abstract**

In this paper, we prove that a bijective map $\phi$ form $\mathscr{A}$, a standard operator algebra on a Banach space of dimension $> 1$, onto a ring that satisfies

$$\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A), \ (A, B \in \mathscr{A})$$

is additive.



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Let $\mathscr{R}$ and $\mathscr{R}'$ be rings. A map $\phi : \mathscr{R} \to \mathscr{R}'$ is called a Jordan map if it satisfies

$$\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a) \tag{1}$$

for all $a, b \in \mathscr{R}$. If $\phi$ is additive and both $\mathscr{R}$ and $\mathscr{R}'$ are algebras, then condition (1) is equivalent to

$$\phi(\tfrac{1}{2}(ab + ba)) = \tfrac{1}{2}(\phi(a)\phi(b) + \phi(b)\phi(a)) \tag{2}$$

for all $a, b \in \mathscr{R}$. Recently, several authors studied additivity of a map that satisfies condition (2) [2,3,7]. In [7], Molnár showed that if $\mathscr{R}$ and $\mathscr{R}'$ are standard operator algebras then a bijective map $\phi : \mathscr{R} \to \mathscr{R}'$ satisfying condition (2) is additive. In the

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same paper, he conjectured that a map satisfying condition (1) is also additive. In this paper, we shall give a positive answer. In fact, we obtain more.

Throughout, $X$ is a Banach space of dimension $> 1$. Denote by $B(X)$ the algebra of all linear bounded operators on $X$. A subalgebra of $B(X)$ is called a standard operator algebra if it contains all finite rank operators in $B(X)$. Our result in this paper is the following.

**Theorem.** *Let $X$ be a Banach space*, $\dim X > 1$, *and let $\mathscr{A} \subset B(X)$ be a standard operator algebra. Let $\mathscr{R}$ be a ring.*

*Suppose $\phi : \mathscr{A} \to \mathscr{R}$ is a bijective map satisfying*

$$\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A) \tag{3}$$

*for all $A$, $B \in \mathscr{A}$. Then $\phi$ is additive. Further, $\phi$ is either a ring isomorphism or a ring anti-isomorphism.*

The main technique we will use is the following argument which will be termed a "standard argument". Suppose, $A$, $B$, $S \in \mathscr{A}$ are such that $\phi(S) = \phi(A) + \phi(B)$. Multiplying this equality by $\phi(T)$ ($T \in \mathscr{A}$) from the left and the right, respectively, we get $\phi(T)\phi(S) = \phi(T)\phi(A) + \phi(T)\phi(B)$ and $\phi(S)\phi(T) = \phi(A)\phi(T) + \phi(B)\phi(T)$. Summing them, we have that

$$\begin{aligned}\phi(T)\phi(S) + \phi(S)\phi(T) = {} & \phi(T)\phi(A) + \phi(A)\phi(T) \\ & + \phi(T)\phi(B) + \phi(B)\phi(T).\end{aligned}$$

It follows from (3) that

$$\phi(ST + TS) = \phi(AT + TA) + \phi(BT + TB).$$

Moreover, if

$$\phi(AT + TA) + \phi(BT + TB) = \phi(AT + TA + BT + TB),$$

then by injectivity of $\phi$, we have that

$$ST + TS = AT + TA + BT + TB.$$

The proof is purely algebraic and will be organized in a series of lemmas. We begin with the following trivial one.

**Lemma 1.** $\phi(0) = 0$.

**Proof.** Since $\phi$ is surjective, we can find an $A \in \mathscr{A}$ such that $\phi(A) = 0$. Therefore $\phi(0) = \phi(0A + A0) = \phi(0)\phi(A) + \phi(A)\phi(0) = 0$. □

In the next several lemmas, let $P_1 \in \mathscr{A}$ be a fixed non-trivial idempotent operator and let $P_2 = I - P_1$, where $I$ is the identity operator on $X$. Set $\mathscr{A}_{ij} = P_i\mathscr{A}P_j$,

$i, j = 1, 2$. Then we may write $\mathscr{A} = \mathscr{A}_{11} \oplus \mathscr{A}_{12} \oplus \mathscr{A}_{21} \oplus \mathscr{A}_{22}$. It should be mentioned that this idea is from Martinadale [6], who studied additivity of multiplicative maps on rings. In what follows, when we write $A_{ij}$, it indicates that $A_{ij} \in \mathscr{A}_{ij}$.

**Lemma 2.** *Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$.*

(i) *For $T_{ij} \in \mathscr{A}_{ij}(1 \leqslant i, j \leqslant 2)$, we have that*

$$T_{ij}S + ST_{ij} = T_{ij}S_{j1} + T_{ij}S_{j2} + S_{1i}T_{ij} + S_{2i}T_{ij}.$$

(ii) *If $T_{ij}S_{jk} = 0$ for every $T_{ij} \in \mathscr{A}_{ij}(1 \leqslant i, j, k \leqslant 2)$, then $S_{jk} = 0$. Dually, if $S_{ki}T_{ij} = 0$ for every $T_{ij} \in \mathscr{A}_{ij}(1 \leqslant i, j, k \leqslant 2)$, then $S_{ki} = 0$.*

(iii) *If $T_{ij}S + ST_{ij} \in \mathscr{A}_{ij}$ for every $T_{ij} \in \mathscr{A}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$, then $S_{ji} = 0$.*

(iv) *If $S_{ii}T_{ii} + T_{ii}S_{ii} = 0$ for every $T_{ii} \in \mathscr{A}_{ii}$ $(i = 1, 2)$, then $S_{ii} = 0$.*

(v) *If $T_{jj}S + ST_{jj} \in \mathscr{A}_{ij}$ for every $T_{ij} \in \mathscr{A}_{jj}$ $(1 \leqslant i \neq j \leqslant 2)$, then $S_{ji} = 0$ and $S_{jj} = 0$. Dually, if $T_{jj}S + ST_{jj} \in \mathscr{A}_{ji}$ for every $T_{jj} \in \mathscr{A}_{ij}$ $(1 \leqslant i \neq j \leqslant 2)$, then $S_{ij} = 0$ and $S_{jj} = 0$.*

**Proof.**

(i) It is an easy computation.

(ii) It is an easy consequence of the fact that $\mathscr{A}$ is prime in the sense that $A\mathscr{A}B = 0$ implies either $A = 0$ or $B = 0$.

(iii) Since $T_{ij}S + ST_{ij} \in \mathscr{A}_{ij}$, we have that $(T_{ij}S + ST_{ij})P_i = 0$. By (i), we see that $T_{ij}S_{ji} = 0$ for every $T_{ij} \in \mathscr{A}_{ij}$. Hence by (ii), $S_{ji} = 0$.

(iv) The condition $S_{ii}T_{ii} + T_{ii}S_{ii} = 0$ is equivalent to $P_iSP_i\mathscr{A}P_i + P_i\mathscr{A}P_iSP_i = 0$. Note that $\mathscr{A}$ is dense in $B(X)$ under the strong operator topology. We can take a net $\{T_\alpha\} \subset \mathscr{A}$ such that SOT-$\lim_\alpha$ $T_\alpha = I$. Taking the limit in $P_iSP_iT_\alpha P_i + P_iT_\alpha P_iSP_i = 0$, we get $2P_iSP_i = 0$. That is $S_{ii} = 0$.

(v) Since $T_{jj}S + ST_{jj} \in \mathscr{A}_{ij}$, we have that $(T_{jj}S + ST_{jj})P_i = 0$. By (i) and (ii), we get that $S_{ji} = 0$.

Since $T_{jj}S + ST_{jj} \in \mathscr{A}_{ij}$ again, we have that $P_j(T_{ij}S + ST_{jj})P_j = 0$. It follows from (i) that $T_{jj}S_{jj} + S_{jj}T_{jj} = 0$ for every $T_{jj} \in \mathscr{A}_{jj}$. We now deduce from (iv) that $S_{jj} = 0$. $\square$

**Lemma 3.** $\phi(A_{ii} + A_{ij}) = \phi(A_{ii}) + \phi(A_{ij}),\ 1 \leqslant i \neq j \leqslant 2.$

**Proof.** Since $\phi$ is surjective, we may find an element $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{ii}) + \phi(A_{ij}). \tag{4}$$

For $T_{ij} \in \mathscr{A}_{ij}$, applying a standard argument to (4), we get

$$\begin{aligned}\phi(T_{ij}S + ST_{ij}) &= \phi(T_{ij}A_{ii} + A_{ii}T_{ij}) + \phi(T_{ij}A_{ij} + A_{ij}T_{ij})\\ &= \phi(A_{ii}T_{ij}) + \phi(0) = \phi(A_{ii}T_{ij}).\end{aligned}$$

Therefore, $T_{ij}S + ST_{ij} = A_{ii}T_{ij}$ for every $T_{ij} \in \mathscr{A}_{ij}$. It follows from Lemma 2(iii) that $S_{ji} = 0$. Hence by Lemma 2(i), we see that

$$T_{ij}S_{jj} = S_{ii}T_{ij} = A_{ii}T_{ij} \quad \forall T_{ij} \in \mathscr{A}_{ij}. \tag{5}$$

For $T_{jj} \in \mathscr{A}_{jj}$, applying a standard argument to (4) again, we have that $T_{ij}S + ST_{jj} = A_{ij}T_{jj}$. It follows from Lemma 2(v) that $S_{ij} = 0$. Hence by Lemma 2(i), we have that $S_{ij}T_{jj} = A_{ij}T_{jj}$ for every $T_{jj} \in \mathscr{A}_{jj}$ and then $S_{ij} = A_{ij}$ by Lemma 2(ii). Moreover from (5) we get that $S_{ii}T_{ij} = A_{ii}T_{ij}$ for every $T_{ij} \in \mathscr{A}_{ij}$. Hence by Lemma 2(ii) we have that $S_{ii} = A_{ii}$. Consequently, $S = A_{ii} + A_{ij}$. We are done. □

Similarly, we have that:

**Lemma 4.** $\phi(A_{ii} + A_{ji}) = \phi(A_{ii}) + \phi(A_{ji}),\ 1 \leqslant j \neq i \leqslant 2.$

**Lemma 5.** $\phi(A_{12} + B_{12}A_{22}) = \phi(A_{12}) + \phi(B_{12}A_{22}).$

**Proof.** Compute

$$\begin{aligned} A_{12} + B_{12}A_{22} &= (P_1 + B_{12})(A_{12} + A_{22}) \\ &= (P_1 + B_{12})(A_{12} + A_{22}) + (A_{12} + A_{22})(P_1 + B_{12}). \end{aligned}$$

Then using (3) and Lemmas 3 and 4, we have that

$$\begin{aligned} \phi(A_{12} + B_{12}A_{22}) &= \phi(P_1 + B_{12})\phi(A_{12} + A_{22}) + \phi(A_{12} + A_{22})\phi(P_1 + B_{12}) \\ &= (\phi(P_1) + \phi(B_{12}))(\phi(A_{12}) + \phi(A_{22})) \\ &\quad + (\phi(A_{12}) + \phi(A_{22}))(\phi(P_1) + \phi(B_{12})) \\ &= (\phi(P_1)\phi(A_{12}) + \phi(A_{12})\phi(P_1)) \\ &\quad + (\phi(B_{12})\phi(A_{22}) + \phi(A_{22})\phi(B_{12})) \\ &\quad + (\phi(P_1)\phi(A_{22}) + \phi(A_{22})\phi(P_1)) \\ &\quad + (\phi(A_{12})\phi(B_{12}) + \phi(B_{12})\phi(A_{12})) \\ &= \phi(P_1A_{12} + A_{12}P_1) + \phi(A_{22}B_{12} + B_{12}A_{22}) \\ &\quad + \phi(P_1A_{22} + A_{22}P_1) + \phi(A_{12}B_{12} + B_{12}A_{12}) \\ &= \phi(A_{12}) + \phi(B_{12}A_{22}). \qquad \square \end{aligned}$$

**Lemma 6.** $\phi(A_{21} + A_{22}B_{21}) = \phi(A_{21}) + \phi(A_{22}B_{21}).$

**Proof.** Compute

$$\begin{aligned} A_{21} + A_{22}B_{21} &= (A_{21} + A_{22})(P_1 + B_{21}) \\ &= (P_1 + B_{21})(A_{21} + A_{22}) + (A_{21} + A_{22})(P_1 + B_{21}). \end{aligned}$$

Now we can complete the proof using a computation similar to that in the proof of Lemma 5. □

**Lemma 7.** *$\phi$ is additive on $\mathscr{A}_{12}$.*

**Proof.** Let $A_{12}, B_{12} \in \mathscr{A}_{12}$ and choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{12}) + \phi(B_{12}). \tag{6}$$

For $T_{22} \in \mathscr{A}_{22}$, applying a standard argument to the equation above and using Lemma 5, we get

$$\phi(T_{22}S + ST_{22}) = \phi(A_{12}T_{22}) + \phi(B_{12}T_{22}) = \phi((A_{12} + B_{12})T_{22}).$$

Hence

$$T_{22}S + ST_{22} = (A_{12} + B_{12})T_{22} \tag{7}$$

for every $T_{22} \in \mathscr{A}_{22}$. It follows from Lemma 2(v) that $S_{22} = S_{21} = 0$. Moreover, by (7) and Lemma 2(i) we have $S_{12}T_{22} = (A_{12} + B_{12})T_{22}$ for every $T_{22} \in \mathscr{A}_{22}$ and then $S_{12} = A_{12} + B_{12}$.

Now there remains to prove that $S_{11} = 0$. For $T_{12} \in \mathscr{A}_{12}$, applying a standard argument to (6) again, we get that $T_{12}S + ST_{12} = 0$. Since we have shown that $S_{22} = S_{21} = 0$, we have that $S_{11}T_{12} = 0$ for every $T_{12} \in \mathscr{A}_{12}$. Hence from Lemma 2(ii) we get that $S_{11} = 0$. □

**Lemma 8.** *$\phi$ is additive on $\mathscr{A}_{21}$.*

**Proof.** Let $A_{21}, B_{21} \in \mathscr{A}_{21}$ and choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{21}) + \phi(B_{21}). \tag{8}$$

For $T_{22} \in \mathscr{A}_{22}$, applying a standard argument to (8) and using Lemma 6, we get

$$T_{22}S + ST_{22} = T_{22}(A_{21} + B_{21}). \tag{9}$$

It follows from Lemma 2(v) that $S_{22} = S_{12} = 0$. Hence Eq. (9) becomes,

$$T_{22}S_{21} = T_{22}(A_{21} + B_{21})$$

for every $T_{22} \in \mathscr{A}_{22}$. This implies that $S_{21} = A_{21} + B_{21}$.

Taking a standard argument into account, we get from (8) that $T_{21}S + ST_{21} = 0$ for every $T_{21} \in \mathscr{A}_{21}$. Since $S_{22} = S_{12} = 0$, it follows that $T_{21}S_{11} = 0$ for every $T_{21} \in \mathscr{A}_{21}$. Hence $S_{11} = 0$. Consequently, $S = A_{21} + B_{21}$. □

**Lemma 9.** *$\phi$ is additive on $\mathscr{A}_{ii}$ $(i = 1, 2)$.*

**Proof.** Let $A_{ii}, B_{ii} \in \mathscr{A}_{ii}$ and choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{ii}) + \phi(B_{ii}). \tag{10}$$

Let $j \neq i$. For $T_{jj} \in \mathscr{A}_{jj}$, by a standard argument to (10), we have that $T_{jj}S + ST_{jj} = 0$. It follows from Lemma 2(v) that $S_{ij} = S_{ji} = S_{ij} = 0$.

Now there remains to prove that $S_{ii} = A_{ii} + B_{ii}$. For $T_{ij} \in \mathscr{A}_{ij}$, applying a standard argument to (10) again, we get

$$\phi(T_{ij}S + ST_{ij}) = \phi(A_{ii}T_{ij}) + \phi(B_{ii}T_{ij}).$$

Hence by Lemmas 7 and 8, we have that

$$T_{ij}S + ST_{ij} = (A_{ii} + B_{ii})T_{ij}$$

for every $T_{ij} \in \mathscr{A}_{ij}$. Since $S_{ij} = S_{ji} = S_{ji} = 0$, it follows that $S_{ii}T_{ij} = (A_{ii} + B_{ii})T_{ij}$ for every $T_{ij} \in \mathscr{A}_{ij}$. Hence by Lemma 2(ii) we have that $S_{ii} = A_{ii} + B_{ii}$. $\square$

**Lemma 10.** *$\phi$ is additive on $P_1\mathscr{A} = \mathscr{A}_{11} + \mathscr{A}_{12}$.*

**Proof.** Let $A_{11}, B_{11} \in \mathscr{A}_{11}$ and let $A_{12}, B_{12} \in \mathscr{A}_{12}$. Then by Lemmas 3, 7, and 9 we see that

$$\begin{aligned}\phi((A_{11} + A_{12}) + (B_{11} + B_{12})) &= \phi((A_{11} + B_{11}) + (A_{12} + B_{12})) \\ &= \phi(A_{11} + B_{11}) + \phi(A_{12} + B_{12}) \\ &= \phi(A_{11}) + \phi(B_{11}) + \phi(A_{12}) + \phi(B_{12}) \\ &= \phi(A_{11} + A_{12}) + \phi(B_{11} + B_{12}). \qquad \square\end{aligned}$$

**Lemma 11.** $\phi(A_{11} + A_{22}) = \phi(A_{11}) + \phi(A_{22})$.

**Proof.** Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{11}) + \phi(A_{22}). \tag{11}$$

Then applying a standard argument to (11) we have that

$$P_1S + SP_1 = A_{11}P_1 + P_1A_{11} = 2A_{11}.$$

By a simple computation, we get that $S_{12} = S_{21} = 0$ and $S_{11} = A_{11}$. Furthermore, a standard argument to (11) yields that

$$\phi(T_{22}S + ST_{22}) = \phi(T_{22}A_{22} + A_{22}T_{22})$$

for every $T_{22} \in \mathscr{A}_{22}$. Hence by the bijectivity of $\phi$, we have that

$$T_{22}S + ST_{22} = T_{22}A_{22} + A_{22}T_{22}$$

for every $T_{22} \in \mathscr{A}_{22}$. Since $T_{22}S = T_{22}S_{22}$ and $ST_{22} = S_{22}T_{22}$, it follows that $T_{22}S_{22} + S_{22}T_{22} = T_{22}A_{22} + A_{22}T_{22}$. That is,

$$(S_{22} - A_{22})T_{22} + T_{22}(S_{22} - A_{22}) = 0$$

for every $T_{22} \in \mathscr{A}_{22}$. Hence $S_{22} - A_{22} = 0$ by Lemma 2(iv). Consequently $S = A_{11} + A_{22}$. $\square$

**Lemma 12.** $\phi(A_{12} + A_{21}) = \phi(A_{12}) + \phi(A_{21})$.

**Proof.** Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{12}) + \phi(A_{21}). \tag{12}$$

For $T_{21} \in \mathscr{A}_{12}$, applying a standard argument to (12), we have that

$$\begin{aligned}\phi(T_{12}S + ST_{12}) &= \phi(A_{12}T_{12} + T_{12}A_{12}) + \phi(A_{21}T_{12} + T_{12}A_{21}) \\ &= \phi(A_{21}T_{12} + T_{12}A_{21}).\end{aligned}$$

Hence by injectivity of $\phi$, we have that

$$T_{12}S + ST_{12} = A_{21}T_{12} + T_{12}A_{21}$$

for every $T_{12} \in \mathscr{A}_{12}$. Multiplying this equality by $P_1$ from the right, we get that $T_{12}S_{21} = T_{21}A_{21}$ for every $T_{12} \in \mathscr{A}_{12}$. It follows from Lemma 2(ii) that $S_{21} = A_{21}$. Hence by Lemma 2(i), we have that

$$T_{12}S_{22} + S_{11}T_{12} = 0 \tag{13}$$

for every $T_{12} \in \mathscr{A}_{12}$. An argument similar to what has led to the equality $S_{21} = A_{21}$ proves that $S_{12} = A_{12}$ also holds.

Applying a standard argument to (12) again, we have that

$$\begin{aligned}\phi(P_1S + SP_1) &= \phi(P_1A_{12} + A_{12}P_1) + \phi(P_1A_{21} + A_{21}P_1) \\ &= \phi(A_{12}) + \phi(A_{21}) = \phi(S).\end{aligned}$$

Therefore $S = P_1S + SP_1$. This implies that $S_{11} = 0$. Hence we deduce from (13) and Lemma 2(ii) that $S_{22} = 0$. Consequently, $S = A_{12} + A_{21}$. $\square$

**Lemma 13.** $\phi(A_{11} + A_{12} + A_{21}) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21})$.

**Proof.** Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that $\phi(S) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21})$. Then by Lemmas 3 and 4, we have that

$$\phi(S) = \phi(A_{11} + A_{12}) + \phi(A_{21}), \tag{14}$$

$$\phi(S) = \phi(A_{11} + A_{21}) + \phi(A_{21}). \tag{15}$$

For $T_{21} \in \mathscr{A}_{21}$, applying a standard argument to (14) we have that

$$T_{21}S + ST_{21} = T_{21}A_{11} + A_{12}T_{21} + T_{21}A_{12}. \tag{16}$$

Multiplying this equality by $P_1$ from the left, we get that $S_{12}T_{21} = A_{12}T_{21}$ for every $T_{21} \in \mathscr{A}_{21}$. It follows that $S_{12} = A_{12}$. Similarly, for $T_{12} \in \mathscr{A}_{12}$, applying a standard argument to (15), we get that $S_{21} = A_{21}$. Multiplying (16) by $P_2$ and $P_1$ from the left and from the right, we get that

$$T_{21}S_{11} + S_{22}T_{21} = T_{21}A_{11} \tag{17}$$

for every $T_{21} \in \mathscr{A}_{21}$.

For $T_{22} \in \mathscr{A}_{22}$, applying a standard argument to (14), we see that

$$\phi(T_{22}S + ST_{22}) = \phi(A_{12}T_{22}) + \phi(T_{22}A_{21}) = \phi(A_{12}T_{22} + T_{22}A_{21}),$$

making a use of Lemma 12. Therefore,

$$T_{22}S + ST_{22} = A_{12}T_{22} + T_{22}A_{21}$$

for every $T_{22} \in \mathscr{A}_{22}$. Multiplying this equality by $P_2$ from the left and from the right, we get that $T_{22}S_{22} + S_{22}T_{22} = 0$ for every $T_{22} \in \mathscr{A}_{22}$. It follows from Lemma 2(iv) that $S_{22} = 0$. Hence, from (17), we get $S_{11} = A_{11}$. Consequently, $S = A_{11} + A_{12} + A_{21}$. $\square$

**Lemma 14.** $\phi(A_{11} + A_{12} + A_{21} + A_{22}) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22})$.

**Proof.** Choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathscr{A}$ such that

$$\phi(S) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22}). \tag{18}$$

Then we have

$$\begin{aligned}\phi(P_1S + SP_1) &= \phi(P_1)\phi(S) + \phi(S)\phi(P_1) \\ &= \phi(P_1)(\phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22})) \\ &\quad + (\phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22}))\phi(P_1) \\ &= \phi(2A_{11}) + \phi(A_{12}) + \phi(A_{21}) \\ &= \phi(2A_{11} + A_{12} + A_{21})\end{aligned}$$

making a use of Lemma 13 in the last equality. It follows that $P_1S + SP_1 = 2A_{11} + A_{12} + A_{21}$. By a simple computation, we get that $S_{11} = A_{11}$, and $S_{21} = A_{21}$.

For $T_{12} \in \mathscr{A}_{12}$, applying a standard argument to (18) again, we have that

$$\phi(T_{12}S + ST_{12}) = \phi(A_{11}T_{12}) + \phi(T_{12}A_{21} + A_{21}T_{12}) + \phi(T_{12}A_{22}).$$

Further, applying a standard argument to the above equality, we have that

$$\begin{aligned}\phi(P_1T_{12}S + T_{12}SP_1 + P_1ST_{12}) &= \phi(A_{11}T_{12}) + \phi(2T_{12}A_{21}) + \phi(T_{12}A_{22}) \\ &= \phi(A_{11}T_{12} + 2T_{12}A_{21} + T_{12}A_{22})\end{aligned}$$

making a use of Lemma 10. Hence we have that

$$T_{12}S_{21} + T_{12}S_{22} + T_{12}S_{21} + S_{11}T_{12} = A_{11}T_{12} + 2T_{12}A_{21} + T_{12}A_{22}.$$

Since we have shown that $S_{11} = A_{11}$ and $S_{21} = A_{21}$, it follows that $T_{12}S_{22} = T_{12}A_{22}$ for every $T_{12} \in \mathscr{A}_{12}$ and hence $S_{22} = A_{22}$. Consequently, $S = A_{11} + A_{12} + A_{21} + A_{22}$. $\square$

**Proof of Theorem.** Let $A + A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$ be in $\mathscr{A}$. Then lemmas are all used in seeing the equalities

$$
\begin{aligned}
&\phi(A+B)\\
&\quad=\phi((A_{11}+B_{11})+(A_{12}+B_{12})+(A_{21}+B_{21})+(A_{22}+B_{22}))\\
&\quad=\phi(A_{11}+B_{11})+\phi(A_{12}+B_{12})+\phi(A_{21}+B_{21})+\phi(A_{22}+B_{22})\\
&\quad=\phi(A_{11})+\phi(B_{11})+\phi(A_{12})+\phi(B_{12})+\phi(A_{21})+\phi(B_{21})+\phi(A_{22})+\phi(B_{22})\\
&\quad=\phi(A_{11}+A_{12}+A_{21}+A_{22})+\phi(B_{11}+B_{12}+B_{21}+B_{22})\\
&\quad=\phi(A)+\phi(B)
\end{aligned}
$$

hold true. That is, $\phi$ is additive.

Now we can complete the proof using a result concerning Jordan isomomorphisms of prime rings [1]. □

Finally we remark that after a careful examination of the argument one can find only properties of a standard operator algebra stated in Lemma 2 are needed. So using the argument presented, one can establish more purely algebraic results. For example, if $\mathscr{R}$ is a unital prime ring which contains a non-trivial idempotent then every a bijective Jordan map from $\mathscr{R}$ onto an arbitrary ring is automatically additive. In the continuation [5] of the present paper, we will deal with more general Jordan maps, that is, (1) is replaced by

$$\phi\left(\frac{1}{k}(ab+ba)\right)=\frac{1}{k}(\phi(a)\phi(b)+\phi(b)\phi(a)),$$

where $k$ is a rational number.

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